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A TOPOLOGICAL APPROACH TO THE REPULSIVE CENTRAL MOTION PROBLEM

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Abstract:
We look for periodic solutions to nonlinear second order system of equations motivated by the Central Motion Problem. We study the repulsive case, the Coulomb problem of a charge being repelled by a source.

Using topological degree methods, we prove that either the problem has a classical solution, or else there exists a family of solutions of perturbed problems that converges uniformly and weakly in H^1 to some limit function u. Furthermore, under appropriate conditions we prove that u is a classical solution. We generalize this results for nonlinearities with a repulsive type singularity.

Keywords: *resonant problems; degree theory; periodic system*
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1 INTRODUCTION

Let us firstly recall the T-periodic Perturbed Central Motion Problem in $\mathbb{R}^N$:

$$\begin{cases} u'' \mp \frac{u}{|u|^3} &= p(t) \qquad t \in \mathbb{R} \\ u(t+T) &= u(t) \qquad t \in \mathbb{R} \end{cases} \tag{1}$$

where $u : \mathbb{R} \to \mathbb{R}^N$. We shall assume that the perturbation p has null average, that is $\overline{p} := \frac{1}{T}\int_0^T p(t)dt = 0$, and that p is $T-$periodic, namely $p(t+T) = p(t)$.

The $\mp$ sign leads to two essentially different physical problems; we shall focus on the '$-$' sign, which corresponds to the repulsive case, when the nonlinearity is $g = -\frac{u}{|u|^3}$. This is the case of the electrostatic Coulomb Central Motion Problem with a charge being repelled by the source. The repulsive problem was broadly studied in the 80' by the Italian school by Solimini [9, 13], Ambrosetti [1] and Coti Zelati [6], among others.

There exists a vast bibliography on this kind of dynamical systems. Lazer and Solimini [9] have considered the scalar case $N = 1$, with $g(u) \to -\infty$ as $u \to 0$, and $\int_0^1 g(t)dt = -\infty$. Using a result proved by Lazer in [8], it is shown that a necessary and sufficient condition for the existence of a weak solution when $g < 0$ and $p \in L^1([0,T])$, is that $\overline{p} < 0$.

In [13], Solimini studied the case $g = \nabla G$, where the potential G has at zero a singularity of repulsive type: for example, the electrostatic potential between two charges of the same sign. More precisely, it is assumed that $G \in C^1(\mathbb{R}^N \backslash \{0\})$ satisfies $\lim_{|u|\to 0} G(u) = +\infty$ and $g = \nabla G$ is *strictly* repulsive at the origin, namely:

$$\limsup_{u\to 0} \left\langle g(u), \frac{u}{|u|} \right\rangle < 0.$$

Under the additional hypothesis

$$\exists\, \delta > 0 \text{ such that, if } \left| \frac{u}{|u|} - \frac{v}{|v|} \right| < \delta, \text{ then } \langle g(u), v\rangle < 0 \tag{2}$$

the existence is shown of a constant $\eta > 0$ such that if $\|p\|_\infty < \eta$ and $\overline{p} = 0$, then the problem has no classical solution. This includes the case of the repulsive central motion, where $G(u) = \frac{1}{|u|}$.

In the same work, the existence of a solution for $\overline{p} \neq 0$ under weaker assumptions is proved. Also, it is remarked that if $\|p\|_\infty$ is large enough, then condition $\overline{p} = 0$ does not imply that the problem is unsolvable.

This is different from what happens in the case $N = 1$, in which u cannot turn around zero; thus, if the repulsive condition $g(u)u < 0$ is assumed for all $u \neq 0$, then the condition $\overline{p} \neq 0$ is necessary.

In a recent paper, Fonda and Toader [7] made an exhaustive analysis on radially symmetric Keplerian-like systems $u'' + h(t, |u|)u = 0$, where $h : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is T-periodic in t. Using a topological degree approach, the existence of classical T-periodic solutions is studied. This work provides also an excellent survey of the known results on the subject. It is focused in the attractive case, in which the main difficulty consists in avoiding collisions. It is also remarked that, for the repulsive case, the difficulty relies in the case $\overline{p} = 0$.

In general the first works in this area worked with the well-known Habets-Sanchez Strong Force condition, which roughly speaking,means that the potential G behaves as $\frac{1}{|u|^\gamma}$ near the origin, with $\gamma \geq 2$; thus, it is not satisfied by the potential we are interested in.

In [14], Zhang employed topological techniques, as we do in this work, in order to study the T-periodic problem and his result says that if $G(u)$ satisfies the previously mentioned Strong Force condition at the singularity, the existence of periodic solutions can be obtained provided that the potential $G(u)$ is smaller than the first eigenvalue of the corresponding Dirichlet problem at infinity. The same kind of assumptions (Strong Force) are made in a work from Coti Zelati [6] for the repulsive case.

The importance of our results is that they can be applied to nonlinearities that do not satisfy the Strong Force Condition, as for example the Central Motion Problem.

In this work, we attacked the singularities by perturbating the problem with continuous approximations of g. To prove the existence of these perturbed problems we use Theorem 1. The result is based on two previous extensions of a well known Theorem by Nirenberg [11] in which g is asked to have uniform limits at infinity different from 0. On the one hand, a result by Ortega and Ward [12], originally in the context of partial differential equations, which allows g to vanish at infinity. On the other hand, a result by Amster and De Nápoli [3], for a ϕ-laplacian operator, in which the asymptotic condition weakened.

A difficult task was to find uniform bounds to these sequences to ensure the existence of a limit function, candidate to be a solution of the original problem. We accomplished this with Theorem 3. This result gave us the existence of a limit function, and a candidate for a solution for the original problem. With stronger conditions, we were able to prove in Theorem 4 that this candidate was in fact a *generalized solution* of the problem.

Also as a part of this last theorem, we got a strong result for the periodic case: If the nonlinearity g was a gradient ($g = \nabla G$), with $\lim_{u \to} G(u) = +\infty$, which implies a stronger kind of repulsiveness, we proved that the limit function was indeed a classical solution of the problem.

Most of the content of this work comes from the author's PhD thesis and for more details the reader should refer to [4].

2 The Central Motion Problem

Let us start making some simple comments on the already described Central Motion Problem (1). We here state a motivation for this problem, the 2-body repulsive periodic problem:

$$\begin{cases} x'' - \frac{y-x}{|x-y|^3} &= & p_1(t) & t \in \mathbb{R} \\ y'' - \frac{x-y}{|x-y|^3} &= & p_2(t) & t \in \mathbb{R} \\ x(t+T) &= & x(t) & t \in \mathbb{R} \\ y(t+T) &= & y(t) & t \in \mathbb{R} \end{cases} \tag{3}$$

with $p_1, p_2 \in C(\mathbb{R}, \mathbb{R}^N)$, and $\overline{p}_1 = \overline{p}_2 = 0$.

Here, $u(t) = (x(t), y(t)) \in C(\mathbb{R}, \mathbb{R}^{2N})$ and the nonlinearity reads

$$g(x, y) = -\frac{1}{|x-y|^3}\,(x - y, y - x)\,, \quad p(t) = (p_1(t), p_2(t)). \tag{4}$$

This is easily transformed into a central motion problem by the change of variables

$$\begin{cases} w = & x - y \\ v = & x + y \\ P = & p_1 + p_2 \\ Q = & p_1 - p_2. \end{cases}$$

Then, we have:

$$\begin{cases} v'' = P(t) \\ w'' - 2\frac{w}{|w|^3} = Q(t). \end{cases}$$

The first equation is easily integrable, and the second one is non other than a variation of problem (1). Topological methods are frequently used to prove existence of solutions of nonlinear systems. Unfortunately, degree theory would not be possible to apply directly without some restrictions, since there are no a-priori bounds for the first equation, namely $v'' = P(t)$ with periodic conditions. In fact, if v is a solution, $v + const$ is also a solution for every constant. Also, an interesting remark is that, besides the singularity of g at 0, it's asymptotic behavior makes it different from the Nirenberg case [11], as the nonlinearity goes to zero at infinity.

The first problem that arises is that when $|x - y|$ goes to zero, g goes to infinity. So we consider continuous perturbations of the nonlinearity. Letting $\varepsilon > 0$, we take a continuous g_ε. Next, we try to avoid the fact that g_ε is zero in the diagonal subspace $\{x = y\}$ of dimension N. We do so by restraining ourselves to the subspace:

$$V = \{u \in C_{per}(\mathbb{R}, \mathbb{R}^{2N}) : \overline{x} + \overline{y} = 0\},$$

with $C_{per}(\mathbb{R}, \mathbb{R}^{2N}) := \{v : \mathbb{R} \to \mathbb{R}^{2N} : v(t) = v(t + T), \forall t \in \mathbb{R}\}$ are the $T-$periodic continuous functions.

Working only in this subspace we attack two problems at once: On one hand we avoid possible collisions. On the other hand, viewing the problem as two different problems after changing variables, we would be able to find a-priori bounds for v, in V. That is somehow the idea behind the degree approach we will use. The perturbation g_ε is carefully defined later on in (11).

The second equation, $w'' - 2\frac{w}{|w|^3} = Q(t)$, lead us to problem (1) taking $u = \frac{w}{2^{3/2}}$. The first difficulty arises on the fact that g is singular at 0; a reasonable way to overcome it consists in considering, for $\varepsilon > 0$, the function $g_\varepsilon(u) = -\frac{u}{\varepsilon + |u|^3}$ and then studying the convergence of the solutions u_ε of the perturbed systems

$$\begin{cases} u'' - \frac{u}{\varepsilon + |u|^3} &= p(t) & t \in \mathbb{R} \\ u(t + T) &= u(t) & t \in \mathbb{R}. \end{cases} \tag{5}$$

The second difficulty relies on the fact that g_ε vanishes at infinity; however, in this case the existence of at least one solution u_ε of (5) for each $\varepsilon > 0$ follows as an immediate consequence of the results obtained for the nonsingular case, studied in [4], which we state here, in a simplified version:

Theorem 1 *Let $p \in C(\mathbb{R}, \mathbb{R}^N)$ be T-periodic such that $\overline{p} = 0$, and let $g \in C(\mathbb{R}^N, \mathbb{R}^N)$ be bounded. Then the nonlinear periodic problem has a solution, provided that (P_1) and (P_2) hold, with:*

(P_1) *There exists a family $\mathcal{F} = \{(U_j, w_j)\}_{j=1,\ldots,K}$ where $\{U_j\}_{j=1,\ldots,K}$ is an open cover of S^{N-1} and $w_j \in S^{N-1}$, such that for some $R_j > 0$ and $j = 1, \ldots, K$:*

$$\langle g(ru), w_j \rangle < 0 \qquad \forall r > R_j \qquad \forall u \in U_j.$$

(P_2) *There exists a constant $R_0 > 0$ such that $\deg(\Phi_r) \neq 0$ for $r \geq R_0$, where $\Phi_r : S^{N-1} \to S^{N-1}$ is given by $\Phi_r(v) := \frac{g(rv)}{|g(rv)|}$.*

Indeed, as

$$\langle g_\varepsilon(u), u\rangle = \left\langle -\frac{u}{\varepsilon+|u|^3}, u\right\rangle = -\frac{|u|^2}{\varepsilon+|u|^3} < 0,$$

for $u \neq 0$, it follows that the boundness condition and (P_2) are trivially satisfied. Moreover, for every $w \in S^{N-1}$ define $U_w = \{u \in S^{N-1} : \langle u, w\rangle > 0\}$. Then $\{U_w\}_w$ covers S^{N-1}, and clearly $\langle g(ru), w\rangle < 0$ for $u \in U_w$ and $r > 0$.

Continuing with the Central Motion Problem, the following computations provide some information concerning the behavior of the family $\{u_\varepsilon\}_\varepsilon$ as $\varepsilon \to 0$:

Multiplying in L^2 the equation in (5) by u_ε, we have:

$$\langle u_\varepsilon'', u_\varepsilon\rangle - \left\langle \frac{u_\varepsilon}{\varepsilon+|u_\varepsilon|^3}, u_\varepsilon\right\rangle = \langle p(t), u_\varepsilon\rangle .$$

Integrating by parts the first term on the left and rearranging the terms we get:

$$\langle u_\varepsilon', u_\varepsilon'\rangle = -\left\langle \frac{u_\varepsilon}{\varepsilon+|u_\varepsilon|^3}, u_\varepsilon\right\rangle - \langle p(t), u_\varepsilon\rangle .$$

Noting that $\left\langle -\frac{u_\varepsilon}{\varepsilon+|u_\varepsilon|^3}, u_\varepsilon\right\rangle \leq 0$, we reach to:

$$\|u_\varepsilon'\|_{L^2}^2 \leq -\langle p(t), u_\varepsilon\rangle .$$

Here, note that $\langle p, u_\varepsilon\rangle = 0$, as $\overline{p} = 0$, so last equation can be written:

$$\|u_\varepsilon'\|_{L^2}^2 \leq -\langle p(t), u_\varepsilon - \overline{u_\varepsilon}\rangle .$$

Finally, taking absolute value we get the bound:

$$\|u_\varepsilon'\|_{L^2}^2 \leq \|p\|_{L^2}\|u_\varepsilon\|_{L^2}.$$

Wirtinger inequality tells us that the following bound also holds:

$$\|u_\varepsilon - \overline{u_\varepsilon}\|_{L^\infty} \leq C\|u_\varepsilon'\|_{L^2}.$$

We then have the following important uniform bounds:

$$\|u_\varepsilon'\|_{L^2} \leq C, \qquad \|u_\varepsilon - \overline{u_\varepsilon}\|_{L^\infty} \leq C \tag{6}$$

where the constant C does not depend on ε. On the other hand, it is easy to prove that the family $\{\overline{u}_\varepsilon\}_\varepsilon \subset \mathbb{R}^N$ is also bounded. Indeed, integrating the main equation in (5) we obtain

$$\int_0^T \frac{u_\varepsilon}{\varepsilon+|u_\varepsilon|^3}dt = 0,$$

and we deduce that

$$-\int_0^T \frac{\overline{u_\varepsilon}}{\varepsilon+|u_\varepsilon|^3}dt = \int_0^T \frac{u_\varepsilon - \overline{u_\varepsilon}}{\varepsilon+|u_\varepsilon|^3}dt.$$

Now, taking norm in $\mathbb{R}^N$:

$$|\overline{u_\varepsilon}| \int_0^T \frac{1}{\varepsilon+|u_\varepsilon|^3}dt \leq \|u_\varepsilon - \overline{u_\varepsilon}\|_{L^\infty} \int_0^T \frac{1}{\varepsilon+|u_\varepsilon|^3}dt.$$

Thus, $|\overline{u_\varepsilon}| \leq C$ for all $\varepsilon > 0$. Hence, for every sequence $\varepsilon_n \to 0$ we may choose a solution $u_n := u_{\varepsilon_n}$ and from the previous bounds there exists a subsequence (still denoted $(u_n)_n$) and a function u such that $u_n \to u$ uniformly and weakly in H^1. Moreover, the following proposition holds:

Proposition 1 *If u is obtained as before and $u \neq 0$ over an open interval I, then $u'' - \frac{u}{|u|^3} = p$ in I, in the classical sense.*

Our last problem concerns the study of the set of zeros of the limit function u. It can be seen that in the central motion problem, if $u \not\equiv 0$ then the zero set is empty, i.e. u is a classical solution. This results will be generalized in the next section.

3 A GENERAL REPULSIVE NONLINEARITY

With problem (1) in mind, we state the more general problem for a function $u : \mathbb{R} \to \mathbb{R}^N$:

$$\begin{cases} u'' + g(u) &= p(t) \qquad t \in \mathbb{R} \\ u(t+T) &= u(t) \qquad t \in \mathbb{R} \end{cases} \tag{7}$$

where $p \in C(\mathbb{R}, \mathbb{R}^N)$ is T-periodic, $\overline{p} = 0$, and $g \in C(\mathbb{R}^N \backslash \{0\}, \mathbb{R}^N)$ has a repulsive type singularity at $u = 0$. By this we mean:

Definition 1 *The function $g \in C(\mathbb{R}^N \backslash \{0\}, \mathbb{R}^N)$ is said to be repulsive at the origin if, for some $\kappa > 0$*

$$\langle g(u), u \rangle < 0 \qquad \text{for } \ 0 < |u| < \kappa. \tag{8}$$

If, furthermore

$$\limsup_{u \to 0} \left\langle g(u), \frac{u}{|u|} \right\rangle := -c, \tag{9}$$

with c a positive constant, then g shall be called strictly repulsive *at the origin.*

We shall proceed in two steps. Firstly, given $\varepsilon > 0$ we introduce the approximated problem:

$$\begin{cases} u'' + g_\varepsilon(u) &= p(t) \qquad t \in \mathbb{R} \\ u(t+T) &= u(t) \qquad t \in \mathbb{R}, \end{cases} \tag{10}$$

where g_ε is a continuous (nonsingular) perturbation of g, and obtain sufficient conditions for the existence of a family of solutions $\{u_\varepsilon\}_\varepsilon$.

We work mainly with approximations such that $g_\varepsilon \to g$ uniformly over compact subsets of $\mathbb{R}^N \backslash \{0\}$ as $\varepsilon \to 0$. We call these admissible.

Secondly, we study the convergence of particular sequences $(u_{\varepsilon_n})_n$ as $\varepsilon_n \to 0$, and study some properties of the limit function u. If $u \not\equiv 0$, then it shall be defined as a *generalized solution* of the problem:

Definition 2 *A function $u \in H^1_{per}(\mathbb{R}, \mathbb{R}^N)$ is said to be a generalized solution of (7) if $u \not\equiv 0$, and for some admissible choice of g_ε there exists a sequence $\varepsilon_n \to 0$ and $(u_{\varepsilon_n})_n$ solutions of (10) for ε_n such that $u_{\varepsilon_n} \to u$ uniformly and weakly in H^1.*

We shall consider the following choice of of g_ε:

$$g_\varepsilon(u) = \begin{cases} g(u) & |u| \geq \varepsilon \\ \rho_\varepsilon(|u|) g\Big(\varepsilon \frac{u}{|u|}\Big) & 0 < |u| < \varepsilon \\ 0 & u = 0, \end{cases} \tag{11}$$

where $\rho_\varepsilon \in C([0, \varepsilon], [0, +\infty))$ is continuous and satisfies $\rho_\varepsilon(0) = 0, \rho_\varepsilon(\varepsilon) = 1$

In particular, for problem (1), taking $\rho_\varepsilon(s) = \frac{s}{\varepsilon}$ the expression simply reduces to $g_\varepsilon(u) = -\frac{u}{(\max\{|u|, \varepsilon\})^3}$.

As we shall see (Proposition 4 below), under the assumption that $G(u) \to +\infty$ as $u \to 0$, both generalized and collision solutions are in fact classical. Conversely, taking g_ε as in (11), it is clear that classical solutions are also generalized solutions.

4 THE APPROXIMATION SCHEME

We begging stating some propositions concerning the properties of those functions defined as the limit of a sequence of perturbed problems. For a proof of them an more details refer to [4]. For convenience, from now on we shall adopt the following notation: $u_n := u_{\varepsilon_n}$, and $g_n := g_{\varepsilon_n}$.

Proposition 2 *Let $(u_n)_n$ and u be defined as before, and assume that $u \neq 0$ over an open interval I. Then u satisfies $u'' + g(u) = p(t)\ \forall\, t \in I$ in the classical sense.*

Condition (9) is the same as in Solimini [13] for the case $g = \nabla G$. It is observed that it does not imply the Habets Sanchez strong force condition. In particular, for any value of $\gamma > -1$ the nonlinearity $g(u) = \frac{-u}{|u|^{\gamma+2}}$ is strictly repulsive, with $c = +\infty$.

In such a situation, it can be proved that the boundary of the set of zeros of the limit function u is discrete; more generally:

Proposition 3 *Let $(u_n)_n$ and u be defined as before, and assume that g is strictly repulsive at the origin. Then the boundary of the set defined by $Z = \{t \in [0,T] : u(t) = 0\}$ is finite, provided that $\|p\|_{L^\infty} < c$, with $c \in (0, +\infty]$ as in (9).*

The following result improves Proposition 3 for the variational case studied in [13]. However, we do not make use of the variational structure of the problem: more generally, it may be assumed that $g = \nabla G$ only near the origin.

Proposition 4 *Assume there exists a neighborhood U of the origin and a function $G \in C^1(U\backslash\{0\}, \mathbb{R})$ such that $g = \nabla G$ on $U\backslash\{0\}$. Further, assume that*

$$\lim_{|u|\to 0} G(u) = +\infty.$$

Then every generalized solution of (7) is classical.

It is worth noting that in this context the repulsive condition (8) implies that $G(u)$ increases when u moves on rays that point towards the origin. However, this specific condition was not necessary in the preceding result, which only uses the fact that $G(0) = +\infty$, since it is not required for the proof of Proposition 2.

Proposition 4 can be regarded as an alternative, in the following way: for g satisfying the assumption, if a sequence $(u_n)_n$ of solutions of (10) for $\varepsilon = \varepsilon_n \to 0$ converges uniformly and weakly in H^1 to some function u, then either $u \equiv 0$, or u is a classical solution of the problem.

It is worth seeing that both situations may occur. For instance, we may consider the following nonlinearity: $g(u) = \frac{u}{|u|^{\gamma+2}}$, with $\gamma \geq 0$. If $p \equiv 0$, then there are no generalized solutions, since they should be classical, because of Proposition 2. In some sense, this is expectable since if g_{ε_n} is given as in (11), then $u_\varepsilon \equiv 0$ is the unique solution of the perturbed problem. On the other hand, for $N = 2$ we may consider $p(t) = -\lambda(\cos(\omega t), \sin(\omega t))$ with $\omega = \frac{2\pi}{T}$, and the circular solution given by $u(t) = r(\cos(\omega t), \sin(\omega t))$, where $\lambda = r\omega^2 + \frac{1}{r^{\gamma+1}}$. After a simple computation, we conclude that the problem has classical solutions for $\lambda \geq (\gamma+2)\left(\frac{\omega^2}{(\gamma+1)}\right)^{\frac{\gamma+1}{\gamma+2}}$.

5 MAIN RESULTS

We have now the tools to state our main results. Note that from now onwards, when we refer g to be *bounded away from the origin* it means that $g \in L^\infty(\mathbb{R}^N\backslash B_1(0))$.

Theorem 2 *Let $g \in C(\mathbb{R}^N\backslash\{0\}, \mathbb{R}^N)$ and assume that (8) holds. Further, assume that g is bounded away form the origin. Then either problem (7) has a classical solution, or else for every sequence $(u_n)_n$ of solutions of (10) with $\varepsilon_n \to 0$ and g_n as in (11), there exists a subsequence that converges uniformly and weakly in H^1.*

We here give a scetch of the proof: If the problem has a classical solution, then there is nothing to prove. Next, assume that (10) admits no classical solutions, and let u_n be a T-periodic solution of

$$u_n'' + g_n(u_n) = p(t),$$

by multiplying by $u_n - \overline{u}_n$ and integrating, we reach to the following inequality:

$$\|u_n'\|_{L^2}^2 \leq \|p\|_{L^2}\|u_n - \overline{u}_n\|_{L^2} + \int_0^T \langle g_n(u_n), u_n - \overline{u_n}\rangle dt. \tag{12}$$

The main idea is to split the integral in two terms:

$$\int_{\{|u_n|>\kappa\}} \langle g_n(u_n), u_n - \overline{u_n}\rangle dt + \int_{\{|u_n|\leq\kappa\}} \langle g_n(u_n), u_n - \overline{u_n}\rangle dt,$$

with κ given by the repulsiveness in (8).

For the first term, we use the definition of g_n, an extension of g. For the second, we use the repulsiveness. Gathering all together:

$$\|u_n'\|_{L^2}^2 \leq C_1\|u_n - \overline{u_n}\|_{L^2} + C_2|\overline{u_n}|.$$

Finally, using Wirtinger's inequality we obtain:

$$\|u_n'\|_{L^2} \leq C|\overline{u_n}|^{\frac{1}{2}}, \qquad \|u_n - \overline{u_n}\|_{L^\infty} \leq C|\overline{u_n}|^{\frac{1}{2}}.$$

At this point we can state that $(\overline{u}_n)_n$ is bounded.

In the previous proof, note that the bounds for $\|u_n\|_{H^1}$ do not depend on the choice of ρ_ε. This is the reason our first main result, with ρ arbitrarily chosen, follows as an immediate consequence of the preceding results:

Theorem 3 *Let $p \in C(\mathbb{R}, \mathbb{R}^N)$ be T-periodic such that $\overline{p} = 0$, and let $g \in C(\mathbb{R}^N\backslash\{0\}, \mathbb{R}^N)$ be repulsive at the origin. Further, assume that g is bounded outside the origin, and that conditions (P_1) and (P_2) hold. Then either (7) has a classical solution, or else for any choice of g_ε as in (11) there exists a sequence $(u_n)_n$ of solutions of problem (10) with $\varepsilon_n \to 0$ that converges uniformly and weakly in H^1.*

Proof.

Given $0 < \varepsilon_n \to 0$ then either $g_n \in C(\mathbb{R}^N, \mathbb{R}^N)$ is bounded for each n. Theorem 1 guarantees the existence of a sequence $(u_n)_n$ of classical solutions of problem (10). Finally, Theorem 2 is applied.

□

The last part of this section is devoted to the second main result, which assumes a different asymptotic condition on g:

Theorem 4 *Let $p \in C(\mathbb{R}, \mathbb{R}^N)$ be T-periodic such that $\overline{p} = 0$, and assume that $g \in C(\mathbb{R}^N\backslash\{0\}, \mathbb{R}^N)$ is repulsive at the origin and bounded outside the origin. Further, assume that condition (P_1) holds, that*

$$\|p\|_{L^\infty} + \sup_{|u|=\tilde{r}} \left\langle g(u), \frac{u}{|u|} \right\rangle < 0 \tag{13}$$

for some $\tilde{r} > 0$ and that the following condition holds:

(P_2') *There exists a constant $R_0 > 0$ such that $\deg(g, B_R(0), 0) \neq (-1)^N$ for $R \geq R_0$,*

then either (7) has a classical solution, or a generalized solution u such that $\|u\|_{L^\infty} \geq \tilde{r}$.

Moreover, if g is strictly repulsive at the origin, then the boundary of the set of zeros of u in $[0, T]$ is finite.

Finally, if $g = \nabla G$ with $\lim_{u\to 0} G(u) = +\infty$, then (7) has a classical solution.

Note that the degree from condition (P_2') is different as the one from (P_2). The first is over a function $g : \mathbb{R}^N \to \mathbb{R}^N$ with respect to a set $(B_R(0) \subset \mathbb{R}^N)$ and a point $(0 \in \mathbb{R}^N)$ and the latter is over a function $\Phi : S^{N-1} \to S^{N-1}$. For a detailed explanation of both definitions and their connection refer to Amster [2] and [4, 5].

The idea of the proof is to adapt the main results form Mawhin's Continuation Theory [10] on an appropriate set $U \in C([0,T], \mathbb{R}^N)$. From Theorem 2, it suffices to show that for each $\varepsilon \leq \tilde{r}$ problem (10) has a solution u_ε such that $\|u_\varepsilon\|_{L^\infty} > \tilde{r}$. To this end, we may follow the general outline of the proof of Theorem 1, but now taking the domain $U = \{u \in C([0,T], \mathbb{R}^N) : \tilde{r} < \|u\|_{L^\infty} < R\}$. Finally it is follows that $\deg(g, \{\tilde{r} < |u| < R\}, 0) \neq 0$, and the conclusion of the Theorem follows.

Condition (P_2') in some sense, says that g is repulsive at ∞, and that it cannot rotate too fast. We have already used the fact that repulsiveness at the origin implies that the Brouwer degree of g_ε over small balls is $(-1)^N$; on the other hand, repulsiveness at ∞ implies that its degree over large balls is 1. Hence, if the assumptions of Theorem 4 are satisfied and g is strictly repulsive at the origin, then there exist generalized solutions for any p continuous and T-periodic such that $\overline{p} = 0$, provided that N is odd.

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References

[1] V. Ambrosetti, A. Coti Zelati. *Periodic Solutions of Singular Lagrangian Systems*. Birkhäuser, Boston, 1993.

[2] P. Amster. *Análisis Nolineal;Métodos Topológicos*. IMPA, Rio de Janeiro, 2009.

[3] P. Amster and P. De Nápoli. Landesman-Lazer type conditions for a system of p-Laplacian like operators. *J. Math. Anal. Appl.*, 326(2):1236–1243, 2007.

[4] P. Amster and M. Maurette. Periodic solutions of systems with singularities of repulsive type. *Adv. Nonlinear Stud.*, 11(1):201–220, 2011.

[5] P. Amster and M. Maurette. An elliptic singular system with nonlocal boundary conditions. *Nonlinear Anal.*, 75(15):5815–5823, 2012.

[6] V. Coti Zelati. Dynamical systems with effective-like potentials. *Nonlinear Anal.*, 12(2):209–222, 1988.

[7] A. Fonda and R. Toader. Periodic orbits of radially symmetric Keplerian-like systems: a topological degree approach. *J. Differential Equations*, 244(12):3235–3264, 2008.

[8] A. C. Lazer. On Schauder's fixed point theorem and forced second-order nonlinear oscillations. *J. Math. Anal. Appl.*, 21:421–425, 1968.

[9] A. C. Lazer and S. Solimini. On periodic solutions of nonlinear differential equations with singularities. *Proc. Amer. Math. Soc.*, 99(1):109–114, 1987.

[10] J. Mawhin. *Topological degree methods in nonlinear boundary value problems*, volume 40 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, R.I., 1979. Expository lectures from the CBMS Regional Conference held at Harvey Mudd College, Claremont, Calif., June 9–15, 1977.

[11] L. Nirenberg. Generalized degree and nonlinear problems. In *Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971)*, pages 1–9. Academic Press, New York, 1971.

[12] R. Ortega and J. R. Ward Jr. A semilinear elliptic system with vanishing nonlinearities. *Discrete Contin. Dyn. Syst.*, (suppl.):688–693, 2003. Dynamical systems and differential equations (Wilmington, NC, 2002).

[13] S. Solimini. On forced dynamical systems with a singularity of repulsive type. *Nonlinear Anal.*, 14(6):489–500, 1990.

[14] M. Zhang. Periodic solutions of damped differential systems with repulsive singular forces. *Proc. Amer. Math. Soc.*, 127(2):401–407, 1999.