MAT

Serie A

Conferencias, seminarios y trabajos de Matemática

ISSN: 1515-4904

5

VI Seminario sobre Problemas de Frontera Libre y sus Aplicaciones.

Tercera Parte

Departamento de Matemática, Rosario, Argentina 2001

UNIVERSIDAD AUSTRAL
FACULTAD DE CIENCIAS EMPRESARIALES

ISSN 1515-4904

MAT

SERIE A : CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMÁTICA

No. 5

VI SEMINARIO SOBRE PROBLEMAS DE FRONTERA LIBRE Y SUS APLICACIONES
Tercera Parte

Domingo A. Tarzia (Ed.)

INDICE

Rosario, Octubre 2001

MAT-Serie A, 5 (2001), 37-41

Uniqueness of Solution to a Free Boundary Problem from Combustion with Transport *

C. LEDERMAN, J. L. VAZQUEZ and N. WOLANSKI

Abstract

We describe results on uniqueness and agreement between different kinds of solutions for a free boundary problem of interest in combustion theory, which were presented in a lecture in the *"VI Seminario sobre problemas de frontera libre y sus aplicaciones"*, Rosario, December 1998. For the detailed proofs of these results, we refer the reader to [LVW1].

1 Introduction

The purpose of these notes is to describe some results, which were proven in [LVW1], on a free boundary problem in heat propagation that in classical terms is formulated as follows: find a nonnegative function $u(x,t)$, defined and continuous in $\mathcal{D} \subset \mathbb{R}^N \times (0,T)$, satisfying the equation

$$\Delta u + \sum a_i \, u_{x_i} - u_t = 0$$

in the positivity set $\mathcal{D} \cap \{u > 0\}$. Besides, we assume that the interior boundary of the positivity set, $\mathcal{D} \cap \partial\{u > 0\}$, so-called *free boundary*, is a regular hypersurface on which the following conditions are satisfied

$$u = 0, \quad -\frac{\partial u}{\partial \nu} = \sqrt{2M}.$$

Here M is a positive constant, and ν denotes outward unit spatial normal to the free boundary. In addition, initial and boundary conditions have to be prescribed on the parabolic boundary of $\mathcal{D}$. Thus, if the domain is a space-time cylinder, $\mathcal{D} = \Omega \times (0,T)$, we prescribe initial data at $t = 0$

$$u(x,0) = u_0(x) \quad \text{for } x \in \overline{\Omega},$$

as well as boundary conditions of Dirichlet or Neumann type on the lateral boundary, $\partial\Omega \times (0,T)$. We will refer to this free boundary problem as Problem P.

This free boundary problem arises in several contexts (cf. [V]). The most important motivation to date has come from combustion theory, where it appears as a limit situation in the description of the propagation of premixed equi-diffusional deflagration flames. In this case, u is the limit, as $\varepsilon \to 0$, of solutions u^ε to equation P_ε:

$$\Delta u^\varepsilon + \sum a_i \, u^\varepsilon_{x_i} - u^\varepsilon_t = \beta_\varepsilon(u^\varepsilon),$$

where $u^\varepsilon(x,t) = T_f - T(x,t)$, with T the temperature of the reactive mixture and T_f the flame temperature, so that $T \le T_f$ and $u^\varepsilon \ge 0$. The function $\beta_\varepsilon(u)$ represents the exothermic chemical reaction and is nonnegative and Lipschitz continuous, positive in an interval $(0, \theta_\varepsilon)$ near $u = 0$ and vanishes otherwise (i.e., reaction occurs only in the range $T_f - \theta_\varepsilon < T < T_f$). The parameter $\varepsilon > 0$ is essentially the inverse of the activation energy of the chemical reaction. Finally, the integral $\int \beta_\varepsilon(u)du = M$ is fixed. The vector $(a_1, \cdots, a_N)$ represents the transport velocity of the reactive mixture. For further details on the model see [BuL].

The study of the limit $P_\varepsilon \to P$ as $\varepsilon \to 0$ was proposed in the 30's by Zeldovich and Frank-Kamenetski [ZF] and has been much discussed in the combustion literature. For the elliptic stationary case see [BCN]

* *Mathematics Subject Classifications:* 35K05, 35K60, 80A25.
Key words: Free boundary problem, combustion, heat equation, uniqueness.
C.L. and N.W. were supported by UBA grants TX47, by CONICET grant PIP0660/98, and by grant BID802/OC-AR PICT03-00000-00137.
J.L.V. was supported by DGICYT Project PB94-0153, and by HCM contract CHRX-CT94-0618.

and [LW]. The study of the limit in the general evolution case has been performed in [CV] in the one phase case, and in [CLW1] and [CLW2] in the two-phase case (i.e., with no sign restriction on u).

Problem P admits *classical* solutions only for good data and for small times, since singularities can arise in finite time. Various concepts of generalized solution have been been introduced in the literature. When we perform the approximation process P_ε and pass to the limit $\varepsilon \to 0$ this gives rise to a kind of solutions to problem P, called *limit solutions* (see [CV]). On the other hand, the concept of *viscosity solution* for problem P was introduced in [CLW2].

In [LVW1] we investigate conditions under which the three concepts agree and produce a unique solution.

2 Main results

The purpose of [LVW1] is to contribute to the questions of unique characterization of the solution of the free-boundary problem P and the consistency of the different solution concepts. The results in [LVW1] can be summarized as saying that, under suitable assumptions on the domain, the reaction function β_ε and on the initial and boundary data,

if a classical solution of problem P exists in a certain time interval, then it is at the same time the unique classical solution, the unique limit solution and also the unique viscosity solution in that time interval.

For definiteness we take as spatial domain a cylinder of the form $\Omega = \mathbb{R}\times\Sigma$ with $\Sigma \subset \mathbb{R}^{N-1}$ a smooth domain, or a semi-cylinder, and we put homogeneous Neumann conditions on the lateral boundary $\mathbb{R}\times\partial\Sigma$. We require monotonicity of the initial data in the direction of the cylinder axis. In the family of problems P_ε we assume that the functions β_ε are defined by scaling of a single function $\beta : \mathbb{R} \to \mathbb{R}$ satisfying:

1. β is a Lipschitz continuous function,
2. $\beta > 0$ in $(0,1)$ and $\beta \equiv 0$ otherwise,
3. $\int \beta(s)ds = M$.

We then define $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$.

We write

$$\mathcal{L}u := \Delta u + \sum a_i\, u_{x_i} - u_t,$$

and we assume that the coefficients a_i in the operator are independent of x_1, the direction of the cylinder axis, and belong to $C^{\alpha,\frac{\alpha}{2}}(\overline{\Sigma}\times[0,T])$.

Our results show in particular that there is a unique limit solution independently of the choice of the function β. Moreover, we actually prove that the limit exists for any approximation of the initial datum.

3 Classical and viscosity solutions

In this section we give precise definitions of the classical and viscosity solutions. In the situations considered in [LVW1] a classical solution is a viscosity solution.

Definition 3.1 *Let $Q = \Omega\times(T_1,T_2)$, with $\Omega\subset\mathbb{R}^N$ a domain, be a space-time cylinder. Let v be a continuous function in $\overline{Q}$. Then v is called a classical subsolution (supersolution) to P in Q if $v\geq 0$ in $\overline{Q}$ and*

1. $\mathcal{L}v \geq 0 \quad (\leq 0) \qquad$ *in* $Q\cap\{v>0\}$.
2. $v\in C^1(\overline{\{v>0\}}), \nabla v\in C^{\alpha,\frac{\alpha}{2}}(\overline{\{v>0\}})$.
3. *For any* $(x,t)\in\{v=0\}\cap\partial\{v>0\}$, *we have* $\nabla v^+(x,t)\neq 0$ *and*

$$-\frac{\partial v^+}{\partial\nu}\geq\sqrt{2M}\qquad(\leq\sqrt{2M}),$$

where $\nu := -\frac{\nabla v^+}{|\nabla v^+|}$. *That is,*

$$|\nabla v^+|\geq\sqrt{2M}\qquad(\leq\sqrt{2M}).$$

We say that v is a classical solution to P in Q if it is both a classical subsolution and a classical supersolution to P.

Definition 3.2 *Let $u \in C(\overline{Q})$; u is called a viscosity subsolution (supersolution) to P in Q if $u \geq 0$ in $\overline{Q}$ and, for every space-time subcylinder $Q' \subset Q$ and for every v bounded classical supersolution (subsolution) to P in Q', with $Q' \cap \partial\{v > 0\}$ bounded,*

$$\begin{array}{lll} u \leq v & (u \geq v) & \text{on } \partial_p Q' \text{ and} \\ v > 0 & \text{on } \overline{\{u > 0\}} \cap \partial_p Q' & \\ (u > 0 & \text{on } \overline{\{v > 0\}} \cap \partial_p Q') & \end{array}$$

implies that $u \leq v$ $(u \geq v)$ in Q'.

The function u is called a viscosity solution to P if it is both a viscosity supersolution and a viscosity subsolution to P.

The following result proves the consistency between both concepts of solution.

Proposition 3.1 *(Proposition 2.1 in [LVW1]) If u is a bounded classical supersolution (subsolution) to P in Q with $Q \cap \partial\{u > 0\}$ bounded, then u is a viscosity supersolution (subsolution) to P in Q.*

Definition 3.3 *Let $\Omega \subset \mathbb{R}^N$ be a domain and let $Q = \Omega \times (0, T)$. Let Γ_N be an open C^1 subset of $\partial\Omega$ and let $\partial_N Q = \Gamma_N \times (0, T)$.*

Let $u \in C(\overline{Q})$. We say that u is a viscosity solution to P in Q with $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$, if $u \geq 0$ and there holds:

For every space-time subcylinder $Q' \subset Q$ and for every v bounded classical supersolution (subsolution) to P in Q', with $Q' \cap \partial\{v > 0\}$ bounded, such that $\frac{\partial v}{\partial \eta} = 0$ on $\partial_p Q' \cap \partial_N Q$,

$$\begin{array}{lll} u \leq v & (u \geq v) & \text{on } \partial_p Q' \setminus \partial_N Q \text{ and} \\ v > 0 & \text{on } \overline{\{u > 0\}} \cap \overline{\partial_p Q' \setminus \partial_N Q} & \\ \big(u > 0 & \text{on } \overline{\{v > 0\}} \cap \overline{\partial_p Q' \setminus \partial_N Q}\big) & \end{array}$$

implies that $u \leq v$ $(u \geq v)$ in Q'.

Proposition 3.2 *(Proposition 2.2 in [LVW1]) Let $\Omega = \mathbb{R} \times \Sigma$ (or $(0, +\infty) \times \Sigma$), $Q = \Omega \times (0, T)$ and $\partial_N Q = \mathbb{R} \times \partial\Sigma \times (0, T)$ (or $\partial_N Q = (0, +\infty) \times \partial\Sigma \times (0, T)$).*

Let u be a bounded classical solution to P in Q with $Q \cap \partial\{u > 0\}$ bounded and $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$. Then u is a viscosity solution to P in Q with $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$.

4 Uniqueness of classical and viscosity solutions

The results in this section say that, under suitable assumptions, a classical solution is the unique viscosity solution to the initial and boundary value problem associated to P and, in particular, it is the unique classical solution.

Theorem 4.1 *(Theorem 3.1 in [LVW1]) Let $\Omega = (0, +\infty) \times \Sigma$, $Q = \Omega \times (0, T)$, $\partial_N Q = (0, +\infty) \times \partial\Sigma \times (0, T)$ and $\partial_D Q = \partial_p Q \setminus \partial_N Q$.*

Let u be a bounded classical solution to P in Q with $\frac{\partial u}{\partial \eta} = 0$ on $\partial_N Q$, such that $u\big|_{\partial_D Q}$ has a bounded, nonempty free boundary and $u_{x_1} < 0$ on $\overline{\{u > 0\}} \cap \partial_D Q$.

Assume that $u(0, x', t) > 0$ for $(x', t) \in \overline{\Sigma} \times [0, T]$ with $u(0, x', t) \in C^{2,1}(\overline{\Sigma} \times [0, T])$.

Let $v \in C(\overline{Q})$ be a viscosity solution to P in Q with $\frac{\partial v}{\partial \eta} = 0$ on $\partial_N Q$.

If $v = u$ on $\partial_D Q$ and $\overline{\{v > 0\}} \cap \partial_D Q = \overline{\{u > 0\}} \cap \partial_D Q$, then $v = u$ in $\overline{Q}$.

For two classical solutions we have the following uniqueness result.

Corollary 4.1 *(Corollary 3.1 in [LVW1]) Let Ω, Q, $\partial_N Q$, $\partial_D Q$ and u as in Theorem 4.1. Let v be a bounded classical solution to P in Q with $\frac{\partial v}{\partial \eta} = 0$ on $\partial_N Q$, such that $v = u$ on $\partial_D Q$. Then, $v = u$ in $\overline{Q}$.*

The next theorem proves the uniqueness of viscosity solution under different assumptions from those in the theorem above. Then, uniqueness of classical solutions follows.

Theorem 4.2 *(Theorem 3.2 in [LVW1]) The result of Theorem 4.1 holds if we let instead $\partial_N Q = \emptyset$ so that $\partial_D Q = \partial_p Q$. Moreover, the result of Theorem 4.1 also holds if we let $\Omega = \mathbb{R} \times \Sigma$ with $\partial_N Q = \mathbb{R} \times \partial\Sigma \times (0, T)$ or $\partial_N Q = \emptyset$, as long as $||u||_{C^{\alpha, \frac{\alpha}{2}}(\overline{Q})} < \infty$. In this case we make no assumptions on u on $\{0\} \times \overline{\Sigma} \times [0, T]$.*

5 Existence and uniqueness of the limit solution

The results in this section say that, under certain assumptions, a classical solution to the initial and boundary value problem associated to P is the uniform limit of any family of solutions to problem P_ε with corresponding boundary data. This in particular implies that such limit exists and is unique. Moreover, it is independent of the choice of the function β.

In particular, under the assumptions of this section our classical solution is the unique classical solution and also the unique viscosity solution.

First, we give the result in a semi-cylinder.

Theorem 5.1 *(Theorem 6.1 in [LVW1]) Let* $\Omega = (0,+\infty)\times\Sigma$, $Q = \Omega\times(0,T)$, $\partial_N Q = (0,\infty)\times\partial\Sigma\times(0,T)$ *and* $\partial_D Q = \partial_p Q \setminus \partial_N Q$.

Let u be a bounded classical solution to P in Q, with $\frac{\partial u}{\partial \eta} = 0$ *on* $\partial_N Q$, *such that* $u\big|_{\partial_D Q}$ *has a bounded, nonempty free boundary and* $u_{x_1} < 0$ *on* $\overline{\{u>0\}} \cap \partial_D Q$.

Assume that $u(0,x',t) > 0$ *for* $(x',t) \in \overline{\Sigma}\times[0,T]$ *with* $u(0,x',t) \in C^{2,1}(\overline{\Sigma}\times[0,T])$.

Let $u^\varepsilon \in C(\overline{Q})$ *with* $\nabla u^\varepsilon \in L^2_{\rm loc}(\overline{Q})$ *be a family of bounded nonnegative weak solutions to* P_ε *in* Q, *with* $\frac{\partial u^\varepsilon}{\partial \eta} = 0$ *on* $\partial_N Q$, *such that* $u^\varepsilon \to u$ *uniformly on* $\partial_D Q$ *and* $\{u^\varepsilon > 0\}\cap\partial_D Q \to \{u>0\}\cap\partial_D Q$. *Then* $u^\varepsilon \to u$ *uniformly in* $\overline{Q}$.

A similar result holds for a full cylinder as spatial domain, under suitable monotonicity assumptions at $x_1 = -\infty$.

Theorem 5.2 *(Theorem 6.2 in [LVW1]) Let* $\Omega = \mathbb{R}\times\Sigma$, $Q = \Omega\times(0,T)$, $\partial_N Q = \mathbb{R}\times\partial\Sigma\times(0,T)$ *and* $\partial_D Q = \partial_p Q\setminus\partial_N Q$.

Let u be a bounded classical solution to P in Q, with $\frac{\partial u}{\partial \eta} = 0$ *on* $\partial_N Q$ *and* $||u||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})} < \infty$, *such that* $u\big|_{\partial_D Q}$ *has a bounded, nonempty free boundary.*

Assume that $u_{x_1} < 0$ *on* $\overline{\{u>0\}}\cap\partial_D Q$ *and* $u_{x_1}(x,0) \le -c_1 e^{c_2 x_1}$ *for* $x_1 \le -a$ *for some constants* c_1, c_2, $a > 0$.

Let $u^\varepsilon \in C(\overline{Q})$ *with* $\nabla u^\varepsilon \in L^2_{\rm loc}(\overline{Q})$ *be a family of bounded nonnegative weak solutions to* P_ε *in* Q, *with* $\frac{\partial u^\varepsilon}{\partial \eta} = 0$ *on* $\partial_N Q$, *such that* $u^\varepsilon \to u$ *uniformly on* $\partial_D Q$, *with* $\{u^\varepsilon>0\}\cap\partial_D Q \to \{u>0\}\cap\partial_D Q$ *and* $|u^\varepsilon(x,0) - u(x,0)| \le k_1\, e^{-k_2\, x_1^2}$ *for* $x_1 \le -a$ *for some constants* k_1, $k_2 > 0$.

Then $u^\varepsilon \to u$ *uniformly in* $\overline{Q}$.

For the proofs of the results in this section, results on existence and regularity of mixed semilinear parabolic problems in non-cylindrical space-time domains were needed. Those results were proven in [LVW3].

6 Extension to the two phase case

The results in [LVW1] were extended to the two phase case. More precisely, in [LVW2] the following two phase free boundary problem is considered: find a function $u(x,t)$, defined in $\mathcal{D}\subset\mathbb{R}^N\times(0,T)$, satisfying

$$\begin{aligned}&\Delta u + \textstyle\sum a_i(x,t)\,u_{x_i} - u_t = 0 \quad \text{in } \{u>0\}\cup\{u<0\},\\ &u = 0\,, \quad |\nabla u^+|^2 - |\nabla u^-|^2 = 2M \quad \text{on } \partial\{u>0\},\end{aligned}$$

where $u^+ = \max(u,0)$, $u^- = \max(-u,0)$, M is a positive constant and a_i are bounded. This is a two phase extension of the free boundary problem considered in the previous sections and we will also refer to this free boundary problem as Problem P.

The purpose of [LVW2] is to investigate conditions under which the three concepts of solution agree and produce a unique solution for the two phase problem. The results of [LVW2] extend those in [LVW1] (eliminating the assumption that $u \ge 0$) and can be summarized as saying that –under appropriate conditions– *if a classical solution of problem P exists in the two phase case, then it is at the same time the unique classical solution, the unique limit solution and also the unique viscosity solution.*

As in [LVW1], the proofs of the results in [LVW2] require results on mixed semilinear parabolic problems in non-cylindrical space-time domains. Those results are extensions of the results in [LVW3] and were proven in [LVW4].

References

[BCN] H. Berestycki, L.A. Caffarelli, L. Nirenberg, *Uniform estimates for regularization of free boundary problems*, “Analysis and Partial Differential Equations”, Lecture Notes in Pure and Applied Mathematics, **122**, Cora Sadosky Ed., Marcel Dekker, New York, (1990), 567–619.

[BuL] J.D. Buckmaster, G.S.S. Ludford, *Theory of Laminar Flames*, Cambridge University Press, Cambridge, 1982.

[CLW1] L.A. Caffarelli, C. Lederman, N. Wolanski, *Uniform estimates and limits for a two phase parabolic singular perturbation problem*, Indiana Univ. Math. J., **46** (2), (1997), 453–490.

[CLW2] L. A. Caffarelli, C. Lederman, N. Wolanski, *Pointwise and viscosity solutions for the limit of a two phase parabolic singular perturbation problem*, Indiana Univ. Math. J., **46** (3) (1997), 719–740.

[CV] L. A. Caffarelli, J. L. Vazquez, *A free boundary problem for the heat equation arising in flame propagation*, Trans. Amer. Math. Soc. **347**, (1995), 411–441.

[LVW1] C. Lederman, J. L. Vazquez, N. Wolanski, *Uniqueness of solution to a free boundary problem from combustion*, Trans. Amer. Math. Soc., **353** (2), (2001), 655–692.

[LVW2] C. Lederman, J. L. Vazquez, N. Wolanski, *Uniqueness in a two-phase free-boundary problem*, Advances in Differ. Equat., **6** (12), (2001), 1409-1442.

[LVW3] C. Lederman, J. L. Vazquez, N. Wolanski, *A mixed semilinear parabolic problem in a noncylindrical space-time domain*, Diff. and Int. Equat., **14** (4), (2001), 385–404.

[LVW4] C. Lederman, J. L. Vazquez, N. Wolanski, *A mixed semilinear parabolic problem from combustion theory*, Electron. Journ. of Diff. Equat., Conf. 06, (2001), 203–214.

[LW] C. Lederman, N. Wolanski, *Viscosity solutions and regularity of the free boundary for the limit of an elliptic two phase singular perturbation problem*, Annali Scuola Norm. Sup. Pisa, Cl. Sci. Serie IV, **27** (2), (1998), 253-288.

[V] J. L. Vazquez, *The free boundary problem for the heat equation with fixed gradient condition*, Free Boundary Problems, Theory and Applications, M. Niezgodka, P. Strzelecki eds., Pitman Research Series in Mathematics, **363**, Longman, 1996, 277–302.

[ZF] Ya.B. Zeldovich, D.A. Frank-Kamenetski, *The theory of thermal propagation of flames*, Zh. Fiz. Khim., **12**, (1938), 100–105 (in Russian); English translation in “Collected Works of Ya.B. Zeldovich”, vol. 1, Princeton Univ. Press, 1992.

Claudia Lederman
Departamento de Matemática, Facultad de Ciencias Exactas
Universidad de Buenos Aires
(1428) Buenos Aires - Argentina
e-mail: clederma@dm.uba.ar

Juan Luis Vazquez
Departamento de Matemáticas, Universidad Autónoma de Madrid
28049 Madrid - Spain
e-mail: juanluis.vazquez@uam.es

Noemi Wolanski
Departamento de Matemática, Facultad de Ciencias Exactas
Universidad de Buenos Aires
(1428) Buenos Aires - Argentina
e-mail: wolanski@dm.uba.ar